Normal ordering for the deformed Heisenberg algebra involving the reflection operator

2005 J. Phys. A: Math. Gen. 38 2305

(http://iopscience.iop.org/0305-4470/38/11/001)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL
J. Phys. A: Math. Gen. **38** (2005) 2305–2310 doi:10.1088/0305-4470/38/11/001

Normal ordering for the deformed Heisenberg algebra involving the reflection operator

Č Burdík[1] and O Navrátil[2]

[1] Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, Trojanova 13, 120 00 Prague 2, Czech Republic
[2] Department of Mathematics, Faculty of Transportation Sciences, Czech Technical University, Na Florenci 25, 110 00 Prague, Czech Republic

Received 19 July 2004, in final form 7 January 2005
Published 2 March 2005
Online at stacks.iop.org/JPhysA/38/2305

Abstract
We give the solution to the problem of normal ordering of monomials $(a^+a)^n$ in pairs of deformed annihilation and creation operators of the Heisenberg algebra involving the reflection operator.

PACS numbers: 02.10.Ox, 05.30.Jp

The normally ordered expansion of an integral power of the number operator a^+a in terms of the boson operators a and a^+ that satisfy the Heisenberg commutation relation $aa^+ = a^+a + 1$ can be written in the form [1]

$$(a^+a)^n = \sum_{k=1}^{n} S_{n,k}(a^+)^k a^k, \tag{1}$$

where the $S_{n,k}$ are the Stirling numbers of the second type

$$S_{n,k} = \frac{1}{k!}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r} r^n.$$

The generalization of the formulae with some applications can be found in many recent papers [2, 3].

In the literature there exist many deformations of different types (Aric–Coon oscillator [4], Macfarlane–Biedenharn oscillator [5, 6], etc).

The analogue of formula (1) for such algebras was carried out in the paper [7] by using a set of deformed Stirling numbers.

A very interesting deformed Heisenberg algebra involving the reflection operator K

$$aa^+ = a^+a + 1 + \nu K, \qquad Ka = -aK, \qquad Ka^+ = -a^+K, \qquad K^2 = 1, \tag{2}$$

where ν is a real parameter, has found many interesting physical applications. This algebra appeared naturally in the context of parafields [8, 9], but earlier it was known in connection with some quantum mechanical systems [10]. Recently, this algebra has been used for the investigation of the quantum mechanical N-body Calogero model [11], for the bosonization of supersymmetric quantum mechanics [12–14] and for describing anyons in (2+1) [14, 15] and (1+1) dimensions [16].

The main goal of this paper is to solve the boson normal ordering problem for this algebra. The result is formulated in the following theorem.

Theorem. *Let $\mathcal{A}$ be associative algebra generated by elements a, a^+ and K which fulfil (2), where ν is a real parameter $\nu \neq \pm 1, \pm 2, \ldots$. Let us define $r_\nu = r + \frac{1-(-1)^r}{2}\nu$ and*

$$[0]_\nu! = 1, \qquad [k+1]_\nu! = (k+1)_\nu \cdot [k]_\nu!. \tag{3}$$

Then, for all $n = 1, 2, \ldots,$ the relations

$$(a^+a)^n = \sum_{k=1}^{n} A_{n,k}(a^+)^k a^k + \sum_{k=1}^{n} B_{n,k}(a^+)^k a^k K, \tag{4}$$

where

$$\begin{aligned} A_{n,k} &= \frac{1}{2}\sum_{r=1}^{k}(-1)^{k-r}\left(\frac{(r_\nu)^n}{[r]_\nu!\cdot[k-r]_{-\nu}!} + \frac{(r_{-\nu})^n}{[r]_{-\nu}!\cdot[k-r]_\nu!}\right) \\ B_{n,k} &= \frac{(-1)^k}{2}\sum_{r=1}^{k}\left(\frac{(r_\nu)^n}{[r]_\nu!\cdot[k-r]_{-\nu}!} - \frac{(r_{-\nu})^n}{[r]_{-\nu}!\cdot[k-r]_\nu!}\right), \end{aligned} \tag{5}$$

hold.

Proof. Formula (4) will be proved by induction. For $n = 1$, the formula is evident from definition. To prove these relations by general n, we use the relation

$$a(a^+)^n = (a^+)^n a + n(a^+)^{n-1} + \frac{1-(-1)^n}{2}\nu(a^+)^{n-1}K, \tag{6}$$

which can easily be verified by induction. If we apply formula (6) to

$$\begin{aligned} (a^+a)^{n+1} &= a^+a(a^+a)^n = a^+a\left(\sum_{k=1}^{n} A_{n,k}(a^+)^k a^k + \sum_{k=1}^{n} B_{n,k}(a^+)^k a^k K\right) \\ &= \sum_{k=1}^{n+1} A_{n+1,k}(a^+)^k a^k + \sum_{k=1}^{n+1} B_{n+1,k}(a^+)^k a^k K, \end{aligned}$$

we obtain for $A_{n,k}$ and $B_{n,k}$ the relations

$$\begin{aligned} A_{n+1,1} &= A_{n,1} - \nu B_{n,1} \\ B_{n+1,1} &= B_{n,1} - \nu A_{n,1} \end{aligned} \tag{7}$$

$$\begin{aligned} A_{n+1,k} &= kA_{n,k} - \frac{1-(-1)^k}{2}\nu B_{n,k} + A_{n,k-1} \\ B_{n+1,k} &= kB_{n,k} - \frac{1-(-1)^k}{2}\nu A_{n,k} + B_{n,k-1} \end{aligned} \tag{8}$$

for $k = 2, 3, \ldots, n$ and

$$A_{n+1,n+1} = A_{n,n} = 1, \qquad B_{n+1,n+1} = B_{n,n} = 0. \tag{9}$$

For $k = 1$, equation (5) reads

$$A_{n,1} = \frac{(1+\nu)^{n-1}}{2} + \frac{(1-\nu)^{n-1}}{2}, \qquad B_{n,1} = -\frac{(1+\nu)^{n-1}}{2} + \frac{(1-\nu)^{n-1}}{2},$$

and it is easy to verify by direct calculation that equations (7) hold for all $n = 1, 2, \ldots$.

The proof of formulae (8) is straightforward by using the relation

$$(-1)^{k-r}k + \frac{1-(-1)^k}{2}\nu - (-1)^{k-r}(k-r)_{-\nu} = (-1)^{k-r}r_\nu,$$

which holds true for any natural r, $1 \leqslant r \leqslant k$.

To prove (9), we denote

$$A_n = 2A_{n,n} = \sum_{r=1}^{n}(-1)^{n-r}\left(\frac{(r_\nu)^n}{[r]_\nu! \cdot [n-r]_{-\nu}!} + \frac{(r_{-\nu})^n}{[r]_{-\nu}! \cdot [n-r]_\nu!}\right)$$

$$B_n = 2(-1)^n B_{n,n} = \sum_{r=1}^{n}\left(\frac{(r_\nu)^n}{[r]_\nu! \cdot [n-r]_{-\nu}!} - \frac{(r_{-\nu})^n}{[r]_{-\nu}! \cdot [n-r]_\nu!}\right).$$

It is easy to show that $A_1 = 2$ and $B_1 = 0$. Next we continue by induction. Let $A_k = 2$ and $B_k = 0$ for any $k = 1, 2, \ldots, n$. Then, since the relation

$$\frac{1}{[r-1]_\nu! \cdot [n-r+1]_{-\nu}!} + \frac{1}{[r]_\nu! \cdot [n-r]_{-\nu}!} = \frac{(n+1)_{(-1)^{r+1}\nu}}{[r]_\nu! \cdot [n-r+1]_{-\nu}!} \tag{10}$$

is fulfilled, the following relations hold:

$$\begin{aligned}
A_{n+1} - 2 = A_{n+1} - A_n &= \sum_{r=1}^{n+1}(-1)^{n-r+1}\left(\frac{(r_\nu)^{n+1}}{[r]_\nu! \cdot [n-r+1]_{-\nu}!} + \frac{(r_{-\nu})^{n+1}}{[r]_{-\nu}! \cdot [n-r+1]_\nu!}\right)\\
&\quad - \sum_{r=1}^{n}(-1)^{n-r}\left(\frac{(r_\nu)^n}{[r]_\nu! \cdot [n-r]_{-\nu}!} + \frac{(r_{-\nu})^n}{[r]_{-\nu}! \cdot [n-r]_\nu!}\right)\\
&= \frac{((n+1)_\nu)^{n+1}}{[n+1]_\nu!} + \frac{((n+1)_{-\nu})^{n+1}}{[n+1]_{-\nu}!} + \sum_{r=1}^{n}(-1)^{n-r+1}\\
&\quad \times \left(\frac{(r_\nu)^n \cdot (n+1)_{(-1)^{r+1}\nu}}{[r]_\nu! \cdot [n-r+1]_{-\nu}!} + \frac{(r_{-\nu})^n \cdot ((n+1)_{(-1)^r\nu})}{[r]_{-\nu}! \cdot [n-r+1]_\nu!}\right)
\end{aligned}$$

$$\begin{aligned}
B_{n+1} = B_{n+1} + B_n &= \sum_{r=1}^{n+1}\left(\frac{(r_\nu)^{n+1}}{[r]_\nu! \cdot [n-r+1]_{-\nu}!} - \frac{(r_{-\nu})^{n+1}}{[r]_{-\nu}! \cdot [n-r+1]_\nu!}\right)\\
&\quad + \sum_{r=1}^{n}\left(\frac{(r_\nu)^n}{[r]_\nu! \cdot [n-r]_{-\nu}!} - \frac{(r_{-\nu})^n}{[r]_{-\nu}! \cdot [n-r]_\nu!}\right) = \frac{((n+1)_\nu)^{n+1}}{[n+1]_\nu!}\\
&\quad - \frac{((n+1)_{-\nu})^{n+1}}{[n+1]_{-\nu}!} + \sum_{r=1}^{n}\left(\frac{(r_\nu)^n \cdot (n+1)_{(-1)^{r+1}\nu}}{[r]_\nu! \cdot [n-r+1]_{-\nu}!} - \frac{(r_{-\nu})^n \cdot ((n+1)_{(-1)^r\nu})}{[r]_{-\nu}! \cdot [n-r+1]_\nu!}\right)
\end{aligned}$$

The term $(n+1)_{(-1)^{r+1}\nu}$ is equal to

$$(n+1)_{(-1)^{r+1}\nu} = \begin{cases} n+1 & \text{for } n \text{ odd}\\ n+1+\nu & \text{for } n \text{ even, } r \text{ odd}\\ n+1-\nu & \text{for } n \text{ even, } r \text{ even.}\end{cases} \tag{11}$$

Therefore, we can write for odd n

$$A_{n+1} - 2 = (n+1)\sum_{r=1}^{n+1}(-1)^{n-r+1}\left(\frac{(r_\nu)^n}{[r]_\nu!\cdot[n-r+1]_{-\nu}!} + \frac{(r_{-\nu})^n}{[r]_{-\nu}!\cdot[n-r+1]_\nu!}\right)$$

$$B_{n+1} = (n+1)\sum_{r=1}^{n+1}\left(\frac{(r_\nu)^n}{[r]_\nu!\cdot[n-r+1]_{-\nu}!} - \frac{(r_{-\nu})^n}{[r]_{-\nu}!\cdot[n-r+1]_\nu!}\right)$$

and for even n, after rearranging, we obtain

$$A_{n+1} - 2 = (n+1)\sum_{r=1}^{n+1}(-1)^{n-r+1}\left(\frac{(r_\nu)^n}{[r]_\nu!\cdot[n-r+1]_{-\nu}!} + \frac{(r_{-\nu})^n}{[r]_{-\nu}!\cdot[n-r+1]_\nu!}\right)$$
$$+\nu\sum_{r=1}^{n+1}\left(\frac{(r_\nu)^n}{[r_]\nu!\cdot[n-r+1]_{-\nu}!} - \frac{(r_{-\nu})^n}{[r]_{-\nu}!\cdot[n-r+1]_\nu!}\right)$$

$$B_{n+1} = (n+1)\sum_{r=1}^{n+1}\left(\frac{(r_\nu)^n}{[r]_\nu!\cdot[n-r+1]_{-\nu}!} - \frac{(r_{-\nu})^n}{[r]_{-\nu}!\cdot[n-r+1]_\nu!}\right)$$
$$+\nu\sum_{r=1}^{n+1}(-1)^{n-r+1}\left(\frac{(r_\nu)^n}{[r]_\nu!\cdot[n-r+1]_{-\nu}!} + \frac{(r_{-\nu})^n}{[r]_{-\nu}!\cdot[n-r+1]_\nu!}\right).$$

Consequently, to prove the relations $A_{n+1} - 2 = B_n = 0$, it is sufficient to prove that these expressions are equal to zero.

It follows from the assumption of induction that if we exchange $n+1$ with n these relations vanish. Due to $\nu \neq \pm 1, \pm 2, \ldots$, these equations are equivalent to

$$\sum_{r=1}^{n}(-1)^{n-r}\left(\frac{(r_\nu)^{n-1}}{[r]_\nu!\cdot[n-r]_{-\nu}!} + \frac{(r_{-\nu})^{n-1}}{[r]_{-\nu}!\cdot[n-r]_\nu!}\right) = 0$$

$$\sum_{r=1}^{n}\left(\frac{(r_\nu)^{n-1}}{[r]_\nu!\cdot[n-r]_{-\nu}!} - \frac{(r_{-\nu})^{n-1}}{[r]_{-\nu}!\cdot[n-r]_\nu!}\right) = 0.$$

Therefore, the relations $A_{n+1} - 2 = B_{n+1} = 0$ are equivalent to

$$\sum_{r=1}^{n+1}(-1)^{n-r+1}\left(\frac{(r_\nu)^n}{[r]_\nu!\cdot[n-r+1]_{-\nu}!} + \frac{(r_{-\nu})^n}{[r]_{-\nu}!\cdot[n-r+1]_\nu!}\right)$$
$$-\sum_{r=1}^{n}(-1)^{n-r}\left(\frac{(r_\nu)^{n-1}}{[r]_\nu!\cdot[n-r]_{-\nu}!} + \frac{(r_{-\nu})^{n-1}}{[r]_{-\nu}!\cdot[n-r]_\nu!}\right) = 0$$

$$\sum_{r=1}^{n+1}\left(\frac{(r_\nu)^n}{[r]_\nu!\cdot[n-r+1]_{-\nu}!} - \frac{(r_{-\nu})^n}{[r]_{-\nu}!\cdot[n-r+1]_\nu!}\right)$$
$$+\sum_{r=1}^{n}\left(\frac{(r_\nu)^{n-1}}{[r]_\nu!\cdot[n-r]_{-\nu}!} - \frac{(r_{-\nu})^{n-1}}{[r]_{-\nu}!\cdot[n-r]_\nu!}\right) = 0.$$

Similarly, we prove that the conditions $A_{n+1} - 2 = B_{n+1} = 0$ are equivalent to

$$\sum_{r=1}^{n+1}(-1)^{n-r+1}\left(\frac{(r_\nu)^{n-1}}{[r]_\nu!\cdot[n-r+1]_{-\nu}!} + \frac{(r_{-\nu})^{n-1}}{[r]_{-\nu}!\cdot[n-r+1]_\nu!}\right) = 0$$

$$\sum_{r=1}^{n+1}\left(\frac{(r_\nu)^{n-1}}{[r]_\nu!\cdot[n-r+1]_{-\nu}!} - \frac{(r_{-\nu})^{n-1}}{[r]_{-\nu}!\cdot[n-r+1]_\nu!}\right) = 0.$$

In this way, we can decrease the power of r_ν and $r_{-\nu}$ in the sums and to prove that the relations $A_{n+1} - 2 = B_{n+1} = 0$ are equivalent to the conditions

$$\sum_{r=1}^{n+1}(-1)^{n-r+1}\left(\frac{r_\nu}{[r]_\nu!\cdot[n-r+1]_{-\nu}!}+\frac{r_{-\nu}}{[r]_{-\nu}!\cdot[n-r+1]_\nu!}\right)$$
$$= \sum_{r=0}^{n}(-1)^{n-r}\left(\frac{1}{[r]_\nu!\cdot[n-r]_{-\nu}!}+\frac{1}{[r]_{-\nu}!\cdot[n-r]_\nu!}\right)=0 \qquad (12)$$

$$\sum_{r=1}^{n+1}\left(\frac{(r_\nu)^{n-1}}{[r]_\nu!\cdot[n-r+1]_{-\nu}!}-\frac{(r_{-\nu})^{n-1}}{[r]_{-\nu}!\cdot[n-r+1]_\nu!}\right)$$
$$= \sum_{r=0}^{n}\left(\frac{1}{[r]_\nu!\cdot[n-r]_{-\nu}!}-\frac{1}{[r]_{-\nu}!\cdot[n-r]_\nu!}\right)=0. \qquad (13)$$

To prove (13), we put in the second term of the sum $n-r \to r$. In the same way, we can prove (12) for n odd. If n in (12) is even, we use (10). In this case, $n_\nu = n$. Therefore, we can write

$$\sum_{r=0}^{n}\frac{(-1)^{n-r}}{[r]_\nu!\cdot[n-r]_{-\nu}!}=\frac{1}{[n]_{-\nu}!}+\frac{1}{[n]_\nu!}+\sum_{r=1}^{n-1}\frac{(-1)^{n-r}}{[r]_\nu!\cdot[n-r]_{-\nu}!}$$
$$=\frac{1}{[n]_{-\nu}!}+\frac{1}{[n]_\nu!}+\frac{1}{n}\sum_{r=1}^{n-1}\left(\frac{(-1)^{n-r}}{[r-1]_\nu!\cdot[n-r]_{-\nu}!}+\frac{(-1)^{n-r}}{[r]_\nu!\cdot[n-r-1]_{-\nu}!}\right)$$
$$=\frac{1}{[n]_{-\nu}!}+\frac{1}{[n]_\nu!}-\frac{1}{n}\sum_{r=0}^{n-2}\frac{(-1)^{n-r}}{[r]_\nu!\cdot[n-r-1]_{-\nu}!}$$
$$+\frac{1}{n}\sum_{r=1}^{n-1}\frac{(-1)^{n-r}}{[r]_\nu!\cdot[n-r-1]_{-\nu}!}=0,$$

where we use $[n]_\nu = n\cdot[n-1]_\nu!$, which holds for n even. □

In this paper, the problem of normal ordering of monomials $(a^+a)^n$ in pairs of deformed annihilation and creation operators of the Heisenberg algebra involving the reflection operator is solved. The calculation of more general formulae for ordering of monomials $((a^+)^s a^r)^n$ is in progress and will soon be finished.

It is a well-known fact that in the case of the standard Heisenberg algebra the corresponding results are related with Hermitian polynomials. The referee has suggested extending the results of this paper to more general polynomials. We have begun the study of this problem. It is one of many possible applications of the results of this paper.

References

[1] Katriel J 1974 Combinatorial aspects of boson algebra *Lett. Nuovo Cimento* **10** 565

[2] Blasiak P, Penson K A and Solomon A I 2003 The general boson normal ordering problem *Phys. Lett.* **309** 198

[3] Blasiak P, Penson K A and Solomon A I 2003 The boson normal ordering problem and generalized Bell numbers *Ann. Comb.* **7** 127

[4] Arik M and Coon D D 1976 *J. Math. Phys.* **17** 524

[5] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581

[6] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873

[7] Katriel J and Kibler M 1992 *J. Phys. A : Math. Gen.* **25** 2683–91

[8] Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Parastatistics* (Tokyo: Tokyo University Press)
[9] Macfarlane A J 1994 Generalized oscillator systems and their parabosonic interpretation *Proc. Int. Workshop on Symmetry Methods in Physics (JINR, Dubna, 1994)* ed A N Sissakian, G S Pogosyan and S I Vinitsky p 319
Macfarlane A J 1994 *J. Math. Phys.* **35** 1054
[10] Yang L M 1951 *Phys. Rev.* **84** 788
[11] Polychronakos A P 1992 *Phys. Rev. Lett.* B **69** 703
Brink L, Hansson T H and Vasiliev M A 1992 *Phys. Lett.* B **286** 109
Brink L, Hansson T H, Konstein S and Vasiliev M A 1993 *Nucl. Phys.* **401** 591
[12] Brzezinski T, Egusquiza I L and Macfarlane A J 1993 *Phys. Lett.* B **311** 202
[13] Plyushchay M S 1996 *Mod. Phys. Lett.* A **11** 397
[14] Plyushchay M S 1996 *Ann. Phys.* **245** 339
[15] Plyushchay M S 1994 *Phys. Lett.* B **320** 91
[16] Aglietti U, Griguolo L, Jackiw R, Pi S-Y and Seminara D 1996 Anyons and chiral solitons on a line *Preprint* hep-th/9606141